Perth Modern School

Semester One Examination, 2015

Question/Answer Booklet

MATHEMATICS SPECIALIST UNIT 1
Section Two: Calculator-assumed

SOLUTIONS

Student Number: In figures

In words ____________________

Your name ____________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time for this section: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer Booklet
Formula Sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
			Total	150	100

Instructions to candidates

1. The rules for the conduct of Western Australian external examinations are detailed in the *Year 12 Information Handbook 2015*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

Section Two: Calculator-assumed (98 Marks)

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 8 (5 marks)

Three vectors are given by $\mathbf{a} = 7\mathbf{i}$, $\mathbf{b} = 6\mathbf{i} + 9\mathbf{j}$ and $\mathbf{c} = x\mathbf{i} - 5\mathbf{j}$.

(a) Use your calculator to determine the angle between $\mathbf{a}$ and $\mathbf{b}$, to the nearest degree. (2 marks)

Using CAS, angle is 56°.

(b) Determine all possible values of x if $\mathbf{a} + \mathbf{c}$ and $\mathbf{b} + \mathbf{c}$ are perpendicular. (3 marks)

$$\begin{bmatrix} 7+x \\ -5 \end{bmatrix} \cdot \begin{bmatrix} 6+x \\ 4 \end{bmatrix} = 0$$

$$x^2 + 13x + 22 = 0$$

$$x = -11, x = -2$$

Question 9 **(8 marks)**

(a) A multiple choice test has twelve questions and each question has three possible choices. If all questions are attempted, in how many ways can the test be answered? (2 marks)

$$3^{12} = 531441$$

(b) A set S contains all the integers between 3 and 102 inclusive. Determine

(i) how many numbers in set S are multiples of 7. (1 mark)

$$102 \div 7 = 14.57... \Rightarrow 14$$

(ii) how many numbers in set S are multiples of 3 or 7. (2 marks)

$$102 \div 3 = 34$$

$$102 \div (3 \times 7) = 4.857... \Rightarrow 4$$

$$14 + 34 - 4 = 44$$

(iii) how many numbers in set S are multiples of either 3 or 7 but not both. (1 mark)

$$14 + 34 - 2(4) = 40$$

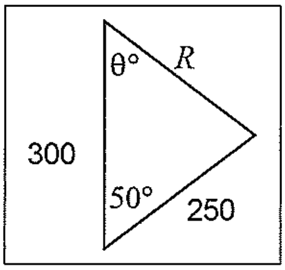
(c) Ten points are equally spaced around the circumference of a circle. Determine the number of simple (non-self-intersecting) convex polygons that can be formed by joining either three, four or five of these points with straight line segments. (2 marks)

$$\begin{aligned} {}^{10}C_3 + {}^{10}C_4 + {}^{10}C_5 &= 120 + 210 + 252 \\ &= 582 \end{aligned}$$

Question 10 **(8 marks)**

Three forces are applied to a body. One has magnitude 300 N and acts due south. Another has magnitude 250 N and acts on a bearing of 050°.

(a) If all three forces are in equilibrium, determine the magnitude and direction of the third force. (4 marks)



$$R^2 = 300^2 + 250^2 - 2 \times 300 \times 250\cos(50)$$
$$R = 236.8161$$

$$\frac{\sin\theta}{250} = \frac{\sin 50}{236.8161}$$
$$\theta = 53.968°$$

$$360 - 54° = 306°$$

Magnitude is 237 N on bearing 306°.

(b) If the third force has a magnitude of 350 N and acts on a bearing of 250°, determine the magnitude and direction of the resultant force. (4 marks)

Due S → 270° from x-axis

050 → 40°

250 → 200°

$$R = 300\begin{bmatrix}\cos(270)\\ \sin(270)\end{bmatrix} + 250\begin{bmatrix}\cos(40)\\ \sin(40)\end{bmatrix} + 350\begin{bmatrix}\cos(200)\\ \sin(200)\end{bmatrix}$$
$$R = \begin{bmatrix}-137.3813\\ -259.0101\end{bmatrix}$$

$$|R| = 293.19$$
$$\angle = -117.94°$$

$$90 + 118 = 208°$$

Magnitude is 293 N on a bearing of 208°.

Question 11 **(6 marks)**

(a) A triangle PQR has vertices $P(1, 1)$, $Q(5, 3)$ and $R(3, 7)$. Determine the vector $\overrightarrow{QM}$, where M is the midpoint of side PR. (3 marks)

$$
\begin{aligned}
QM &= OM - OQ \\
&= OP + \tfrac{1}{2}PR - OQ \\
&= OP + \tfrac{1}{2}(OR - OP) - OQ \\
&= \tfrac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} + \tfrac{1}{2}\begin{bmatrix}3\\7\end{bmatrix} - \begin{bmatrix}5\\3\end{bmatrix} \\
&= \begin{bmatrix}-3\\1\end{bmatrix}
\end{aligned}
$$

(b) ABC is a triangle with point D on side AC such that $AD = \frac{3}{4}AC$. If $\overrightarrow{BA} = \mathbf{a}$ and $\overrightarrow{BD} = \mathbf{d}$, show that $\overrightarrow{BC} = \frac{1}{3}(4\mathbf{d} - \mathbf{a})$. (3 marks)

$$\overrightarrow{AD} = \mathbf{d} - \mathbf{a}$$

$$
\begin{aligned}
\overrightarrow{BC} &= \overrightarrow{BD} + \overrightarrow{DC} \\
&= \overrightarrow{BD} + \tfrac{1}{3}\overrightarrow{AD} \\
&= \mathbf{d} + \tfrac{1}{3}(\mathbf{d} - \mathbf{a}) \\
&= \tfrac{1}{3}(4\mathbf{d} - \mathbf{a})
\end{aligned}
$$

Question 12 **(6 marks)**

(a) Vectors $\mathbf{a}$ and $\mathbf{b}$ have the same magnitude and vectors $\mathbf{a}$ and $\mathbf{c}$ are perpendicular, where $\mathbf{a}=\begin{bmatrix} m \\ n \end{bmatrix}$, $\mathbf{b}=\begin{bmatrix} -4 \\ 6 \end{bmatrix}$ and $\mathbf{c}=\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Determine the values of m and n. (2 marks)

$m^2+n^2=52$ and $2m+3n=0$

Using CAS:

$m=-6, n=4$

$m=6, n=-4$

(b) Determine the scalar projection of a velocity of 12 m/s on a bearing of 65° onto a velocity of 20 m/s on a bearing of 280°, giving your answer to three significant figures. (2 marks)

$$12\times\cos(360-280+65)=12\times\cos(145)$$
$$=-9.8298$$
$$\approx -9.83$$

(c) The work done, in joules, by a force of $\mathbf{F}$ Newtons in changing the displacement of an object by $\mathbf{s}$ metres is given by the scalar product of $\mathbf{F}$ and $\mathbf{s}$.

A force acting on a bearing of 160° does work of 1 200 joules. If the object moved a distance of 350 cm on a bearing of 135°, determine the magnitude of the force. (2 marks)

$$|\mathbf{F}|\times 3.5\times\cos(160-135)=1200$$
$$|\mathbf{F}|=\frac{1200}{3.5\cos(25)}$$
$$|\mathbf{F}|=378.3\text{ N}$$

Question 13 **(8 marks)**

(a) A triangle has vertices at $A(-3,\ 1)$, $B(-1,\ 4)$ and $C(5,\ 0)$.

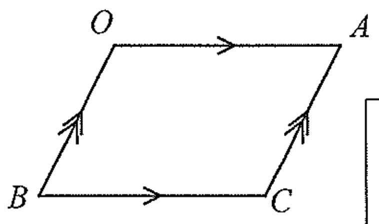
(i) Determine the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{BC}$. (2 marks)

$$AB = \begin{bmatrix} 2 \\ 3 \end{bmatrix},\ AC = \begin{bmatrix} 8 \\ -1 \end{bmatrix},\ BC = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$$

(ii) Use a vector method to prove that triangle ABC is right-angled. (2 marks)

$$AB \bullet BC = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 6 \\ -4 \end{bmatrix}$$
$$= 12 - 12$$
$= 0 \Rightarrow$ triangle right-angled at B

(b) Use a vector method to prove that if the diagonals of a parallelogram are perpendicular to each other, then the parallelogram is a rhombus. (4 marks)



$$OA = \mathbf{a},\ OB = \mathbf{b}$$

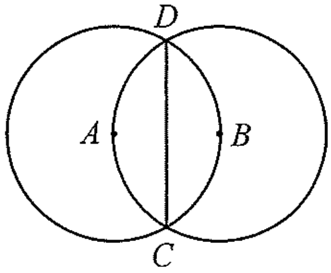
$$OC = \mathbf{a} + \mathbf{b}$$
$$AB = \mathbf{b} - \mathbf{a}$$

If OC & AB are perpendicular

$$OC \bullet AB = 0$$
$$(\mathbf{a} + \mathbf{b}) \bullet (\mathbf{b} - \mathbf{a}) = 0$$
$$\mathbf{a} \bullet \mathbf{b} - \mathbf{a} \bullet \mathbf{a} + \mathbf{b} \bullet \mathbf{b} - \mathbf{b} \bullet \mathbf{a} = 0$$
$$\mathbf{a} \bullet \mathbf{a} = \mathbf{b} \bullet \mathbf{b}$$
$$|\mathbf{a}|^2 = |\mathbf{b}|^2$$
$|\mathbf{a}| = |\mathbf{b}| \Rightarrow$ $OABC$ is a rhombus

Question 14 **(8 marks)**

(a) Two circles of radius 12.6 cm, with centres A and B as shown below, have a common chord CD. Determine, with justification, the length CD. (2 marks)

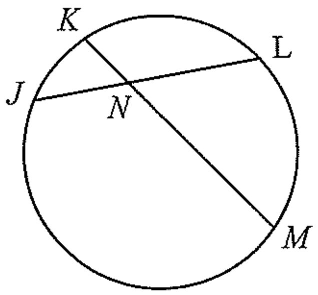


$\sqrt{12.6^2 - 6.3^2} = 10.91$

$10.91 \times 2 = 21.82$

≈ 21.8 cm

(b) In the diagram below, $KN = 10$ cm, $LN = 15$ cm and $MN = 20$ cm. Determine, with justification, the exact length of JN. (3 marks)

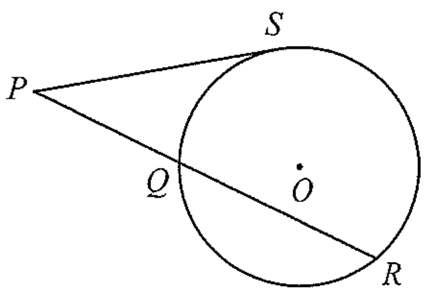


$\Delta JKN \sim \Delta MLN$ (AAA)

$\frac{JN}{20} = \frac{10}{15}$

$JN = 13\frac{1}{3}$

(c) Determine the length PQ, if that the length of chord QR is 10.5 cm and the length of the tangent PS is 9.5 cm. (3 marks)



$PS^2 = PQ \times PR$

$9.5^2 = x(x + 10.5)$

$x = -16.1$ or 5.6

$PQ = 5.6$ cm

Question 15 **(9 marks)**

(a) A small body A has position (12, -3) m relative to another small body B. If a third small body C has position (-5, 6) relative to A, determine the position of B relative to C.

(2 marks)

If A is 'origin' $_BR_C = R_B - R_C$

$$= \begin{bmatrix} -12 \\ 3 \end{bmatrix} - \begin{bmatrix} -5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} -7 \\ -3 \end{bmatrix}$$

(b) To a cyclist moving with velocity (21, -5) km/h the wind appears to have velocity (-9, 3) km/h. Determine the true speed of the wind. (3 marks)

$$_wV_c = V_w - V_c$$

$$\begin{bmatrix} -9 \\ 3 \end{bmatrix} = V_w - \begin{bmatrix} 21 \\ -5 \end{bmatrix}$$

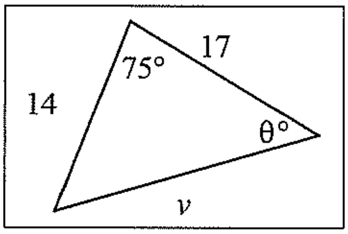
$$V_w = \begin{bmatrix} 12 \\ -2 \end{bmatrix}$$

$$|V_w| = 2\sqrt{37}$$

$$\approx 12.2 \text{ km/h}$$

(c) A small ship is travelling with a constant speed of 14 knots on a bearing of 025° and another, larger ship is travelling with a constant speed of 17 knots on a bearing of 310°.

Determine the velocity of the large ship relative to the small ship. (4 marks)



$$v = {}_LV_S = V_L - V_S$$

$$v^2 = 14^2 + 17^2 - 2(14)(17)\cos(75)$$
$$v = 19.0211$$

$$\frac{\sin\theta}{14} = \frac{\sin 75}{19.0211}$$
$$\theta = 45.312°$$

$$310 - 45 = 265°$$

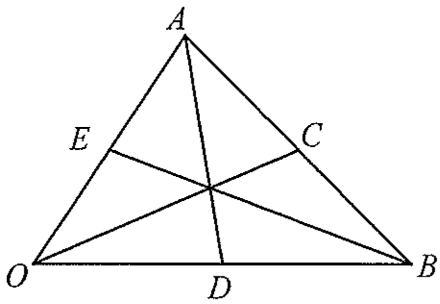
19.02 knots on bearing 265°.

Question 16 **(9 marks)**

The medians of triangle OAB are OC, AD and BE, as shown below.

(A median joins a vertex to the midpoint of the opposite side of the triangle).

Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.



(a) Prove that $\overrightarrow{OC} + \overrightarrow{AD} + \overrightarrow{BE} = \mathbf{0}$. (4 marks)

$$OC = OA + \tfrac{1}{2}AB$$
$$= \tfrac{1}{2}\mathbf{a} + \tfrac{1}{2}\mathbf{b}$$

$$AD = AO + \tfrac{1}{2}OB$$
$$= \tfrac{1}{2}\mathbf{b} - \mathbf{a}$$

$$BE = BO + \tfrac{1}{2}OA$$
$$= \tfrac{1}{2}\mathbf{a} - \mathbf{b}$$

$$OC + AD + BE = \tfrac{1}{2}\mathbf{a} + \tfrac{1}{2}\mathbf{b} + \tfrac{1}{2}\mathbf{b} - \mathbf{a} + \tfrac{1}{2}\mathbf{a} - \mathbf{b}$$
$$= \mathbf{0}$$

(b) The centroid, F, is the point of intersection of the medians.

Determine $\overrightarrow{AF}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. (5 marks)

(*Hint: Let* $\overrightarrow{EF} = h\overrightarrow{EB}$, $\overrightarrow{OF} = k\overrightarrow{OC}$ *and first solve for* h *and* k)

$$AF = AE + EF = AO + OF$$

$$AE + EF = AO + OF$$
$$-\tfrac{1}{2}\mathbf{a} + hEB = -\mathbf{a} + kOC$$
$$-\tfrac{1}{2}\mathbf{a} + h(\mathbf{b} - \tfrac{1}{2}\mathbf{a}) = -\mathbf{a} + k(\tfrac{1}{2}\mathbf{a} + \tfrac{1}{2}\mathbf{b})$$

$$\mathbf{a}: -\tfrac{1}{2} - \tfrac{1}{2}h = -1 + \tfrac{1}{2}k \Rightarrow h + k = 1$$

$$\mathbf{b}: h = \tfrac{1}{2}k$$

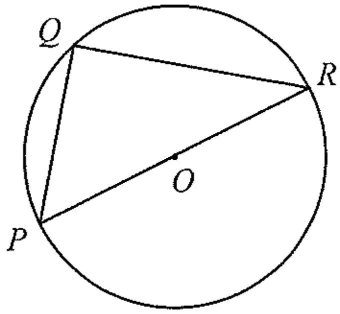
$$h = \tfrac{1}{3},\ k = \tfrac{2}{3}$$

$$AF = -\tfrac{1}{2}\mathbf{a} + \tfrac{1}{3}(\mathbf{b} - \tfrac{1}{2}\mathbf{a})$$
$$= \tfrac{1}{3}\mathbf{b} - \tfrac{2}{3}\mathbf{a}$$

Question 17 **(8 marks)**

(a) The diagram shows a triangle with vertices P, Q and R that lie on a circle with centre O. Chord PR passes through O. Prove, by contradiction, that angle QPR is acute.

(4 marks)



Assume that QPR is not acute,
ie that $\angle QPR \geq 90°$.

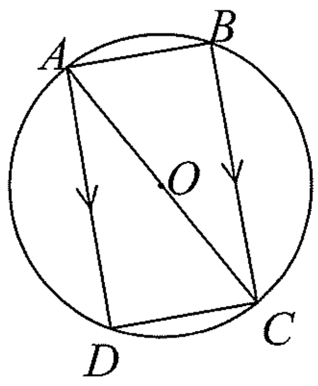
Because PR is a diameter, then $\angle PQR = 90°$.

Because PQR is a triangle, then $\angle QRP > 0°$.

Given that $\angle QPR \geq 90°$, $\angle PQR = 90°$ and $\angle QRP > 0°$, then the sum of the angles in triangle PQR exceeds 180°, but this contradicts the fact that the sum of the angles in a triangle is 180°.

Hence, our assumption that $\angle QPR \geq 90°$ must be false, and so angle QPR is acute.

(b) In the diagram below, O is the centre of the circle on which points A, B, C, and D lie. Chord AC passes through O and BC is parallel to AD. Prove that the quadrilateral $ABCD$ is a rectangle. (4 marks)



$\angle ABC = \angle CDA = 90°$ (Both angles in semi-circle).

$\angle BCA = \angle DAC$ (alternate angles).

AC is common side in $\triangle ABC$ and $\triangle CDA$.

Hence $\triangle ABC$ and $\triangle CDA$ are congruent (ASA).

Hence $AB = CD$ and $BC = DA$.

$\angle BAD = \angle DCB = 90°$ (Co-interior with $\angle ABC$ and $\angle CDA$).

Hence $ABCD$ is a rectangle.

Question 18 **(8 marks)**

(a) A small coach has 24 seats, arranged in six rows of four seats each, with two seats in each row on either side of the central aisle. A group of passengers consisting of ten males and nine females board the bus.

(i) Determine how many combinations of empty seats are possible once everyone has sat down. (1 mark)

$^{24}C_5 = 42504$

(ii) How many fewer combinations are there if the females all sit on one side of the aisle and the males all sit on the other side? (3 marks)

$^{12}C_2 \times {}^{12}C_3 = 14520$

$42504 - 14520 = 27984$ fewer seats

(b) Determine the number of possible four letter permutations of the letters of the word

(i) RELOAD. (1 mark)

$^6P_4 = 360$

(ii) RELOADED. (3 marks)

All different: 360

2 E's, other 2 diff: $^2C_2 \times {}^5C_2 \times \frac{4!}{2!} = 120$

2 D's, other 2 diff: 120

2 E's and 2 D's: $^2C_2 \times {}^2C_2 \times \frac{4!}{2!2!} = 6$

Total: $360 + 120 + 120 + 6 = 606$

Question 19 **(7 marks)**

A small boat has to travel across a river from A to B, where $OA = 60\mathbf{i} + 35\mathbf{j}$ m and $OB = 356\mathbf{i} - 125\mathbf{j}$ m. A uniform current of $-1.5\mathbf{i} + 2.5\mathbf{j}$ m/s is flowing in the river and the boat can maintain a steady speed of 4 m/s.

(a) Determine, in the form $a\mathbf{i} + b\mathbf{j}$, the velocity vector the small boat should set to travel directly from A to B. (5 marks)

$$\overrightarrow{AB} = \begin{bmatrix} 356 \\ -125 \end{bmatrix} - \begin{bmatrix} 60 \\ 35 \end{bmatrix}$$
$$= \begin{bmatrix} 296 \\ -160 \end{bmatrix}$$

Let boat velocity $= \begin{bmatrix} x \\ y \end{bmatrix}$

Then $t\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1.5 \\ 2.5 \end{bmatrix}\right) = \begin{bmatrix} 296 \\ -160 \end{bmatrix} \Rightarrow \dfrac{x - 1.5}{y + 2.5} = \dfrac{296}{-160}$

and $x^2 + y^2 = 4^2$

Solve eqns (CAS) to get

$x = -3.9736, y = 0.4587$ or $x = 2.5604, y = -3.0732$

2nd solns give +ve time $\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2.5604 \\ -3.0732 \end{bmatrix}$

(b) Calculate, to the nearest minute and second, how long the journey will take. (2 marks)

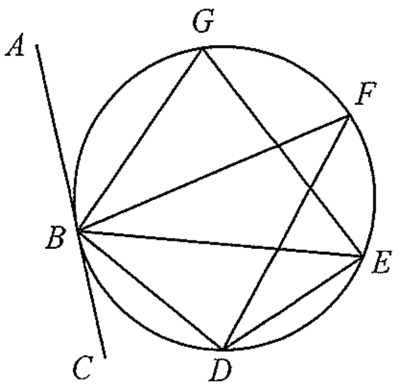
$(2.5604 - 1.5)t = 296$

$t = 279.1$

$\approx$ 4 minutes and 39 seconds

Question 20 **(8 marks)**

(a) In the diagram below, AC is a tangent to the circle at B. If $\angle ABG = 40°$, $\angle GBF = 25°$ and $\angle BFD = 30°$, determine the size of angle DBF. (4 marks)



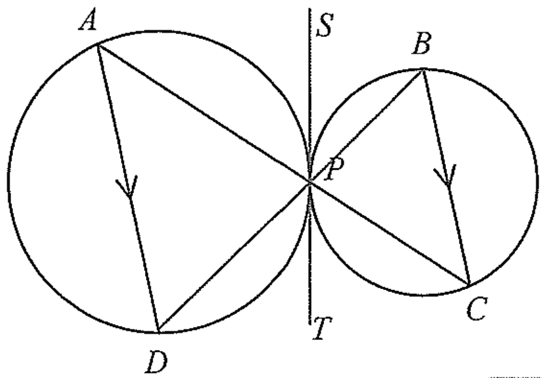
$\angle BEG = \angle ABG = 40$ (alt segment)

$\angle BED = \angle BFD = 30$ (same chord)

$\angle GBD = 180 - \angle GED$ (cyc quad)
$= 180 - 30 - 40$
$= 110$

$\angle DBF = 110 - \angle GBF$
$= 110 - 25$
$= 85°$

(b) In the diagram below, the line AC passes through the point P, where both circles touch each other. The line AD is parallel to line BC. Prove that the points B, P and D are collinear. (4 marks)



Add tangent ST at P. Then
$\angle PCB = \angle BPS$ (Alternate segment theorem)
$\angle PAD = \angle DPT$ (Alternate segment theorem)
But $\angle PCB = \angle PAD$ (Alternate angles)
$\Rightarrow \angle BPS = \angle DPT$
Hence BD is a straight line making vertically opposite angles with AC
$\therefore$ B, P and D are collinear

End of questions

Additional working space

Question number: _________

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